Measure Theory with Ergodic Horizons
Lecture 20

Prop. Let (X,μ) be a measure space and $(f_n), f$ μ-measurable. If $f_n \to_{L^1} f$ then $f_n \to_\mu f$.

Proof. Fix $\alpha > 0$. We need to show that $\delta_\alpha(f_n, f) = \mu(\Delta_\alpha(f_n,f)) \to 0$.
By Chebyschev's inequality, $\mu(\Delta_\alpha(f_n,f)) = \mu(\{x \in X : |f_n(x) - f(x)| \geq \alpha\}) \leq \frac{1}{\alpha} \|f_n - f\|_1$
$\to 0$ as $n \to \infty$. $\square$

Prop (almost Hausdorffness). If $f_n \to_\mu f$ and $f_n \to g$, then $f = g$ a.e.

Proof. Let $\Delta(f,g) := \{x \in X : f \neq g\}$ and note that fixing any $\alpha_n \searrow 0$, we have
$$\Delta(f,g) = \bigcup_{n \in \mathbb{N}} \Delta_{\alpha_n}(f,g),$$
so it is enough to show that $\Delta_\alpha(f,g)$ is null for each $\alpha > 0$. But
$\delta_\alpha(f,g) \leq \delta_{\alpha/2}(f, f_n) + \delta_{\alpha/2}(f_n, g) \to 0$ as $n \to \infty$. $\square$

Def. Call a sequence (f_n) Cauchy in measure if for each $\alpha > 0$,
$$\delta_\alpha(f_n, f_m) \to 0 \quad \text{as} \quad \min(n,m) \to \infty.$$

Prop. (a) If $f_n \to_\mu f$ then (f_n) is Cauchy in measure.
(b) If (f_n) is Cauchy in measure and admits a subsequence $f_{n_k} \to_\mu f$, then $f_n \to_\mu f$.

Proof. HW.

We will show that the converse of part (a) holds, so the convergence in measure "uniformity" is complete.

Theorem. Let (f_n) be Cauchy in measure. Then there is a μ-measurable f such that $f_n \to_\mu f$. Moreover, $f_{n_k} \to f$ a.e. for some subsequence.

Proof. Note that by part (b) above, we may restrict to subsequences.

Claim. We may assume WLOG that $\delta_{2^{-n}}(f_n, f_{n+1}) \leq 2^{-n}$, by restricting to a subsequence.

Proof. We recursively build a subsequence (n_k) such that $\delta_{2^{-k}}(f_{n_k}, f_{n_{k+1}}) \leq 2^{-k}$. Let $n_{-1} := 0$ and choose $n_k > n_{k-1}$ such that $\delta_{2^{-k}}(f_{n_k}, f_m) \leq 2^{-k}$ for all $m \geq n_k$, where such n_k exists by the Cauchy condition. Then the sequence (f_{n_k}) is as desired. $\square$

We now show that for a.e. $x \in X$, $(f_n(x))$ is Cauchy (as a sequence of reals). By Borel-Cantelli, we have that for a.e. $x \in X$ $\exists N$ such that $x \notin B_N := \bigcup_{n \geq N} \Delta_{2^{-n}}(f_n, f_{n+1})$. But that for all $n \geq N$, and $m \geq n$, we have

$$|f_n(x) - f_m(x)| \leq \sum_{i=n}^{m-1} |f_i(x) - f_{i+1}(x)| \leq \sum_{i=n}^{m-1} 2^{-i} \leq \sum_{i=n}^{\infty} 2^{-i} \to 0 \quad \text{as } n \to \infty.$$

By the completeness of $\mathbb{R}$, there is a function $f : X \to \mathbb{R}$ s.t. $f_n(x) \to f(x)$ for a.e. $x \in X$. Hence f is μ-measurable being the ptwise limit of measurable functions. It remains to show that $f_n \to_\mu f$. Let $\alpha > 0$ and let N be large enough so that $\alpha \geq 2^{-N}$. Then for all $n \geq N$,

$$d_\alpha(f_n, f) \leq d_{2^{-n}}(f_n, f) \leq \mu(B_N) \leq \sum_{i=N}^{\infty} 2^{-i} \to 0 \quad \text{as } N \to \infty. \qquad \square$$

Cor. If $f_n \to_\mu f$ then $f_{n_k} \to f$ a.e. for some subsequence. In particular, if $f_n \to_{L^1} f$ then $f_{n_k} \to f$ a.e. for some subsequence.

Proof. By part (a), (f_n) is Cauchy in measure, so by the previous theorem $f_n \to_\mu g$ for some μ-measurable function such that $f_{n_k} \to g$ a.e. for some subsequence. By almost Hausdorffness, $f = g$ a.e. $\square$

This is just the dual to Borel-Cantelli.

$\forall x \exists N \Rightarrow \exists N \forall^{99\%} x$ trick. Let (X, μ) be a finite measure space. Let $P_n \subseteq X$ be an increasing sequence of measurable sets. For every $\delta > 0$,

$$\forall x \in X \; \exists n \in \mathbb{N} \;\; x \in P_n \;\; \Rightarrow \;\; \exists n \; \forall^{-\delta}_x \in X \;\; x \in P_n,$$

where $\forall^{-\delta}_x$ means that the statement is true for all $x \in X$ except for a set of measure $\leq \delta$.

Proof. The hypothesis means that $X = \bigcup_{n \in \mathbb{N}} P_n$ so by monotonicity of measure, P_n has measure $\mu(X) - \delta$ for all large enough $n \in \mathbb{N}$. Hence for such n, we have that $\forall x \in X$ we have $x \in P_n$, except for $x \in X \setminus P_n$. $\square$

Prop. Let (X,μ) be a finite measure space and $(f_n), f$ be μ-measurable functions. If $f_n \to f$ a.e. then $f_n \to_\mu f$.

Proof. Discarding a null set, we may assume $f_n \to f$ so for each $\alpha > 0$, $\forall x \in X \; \exists N_{x,\varepsilon} \; \forall n \geq N_x \; |f_n(x) - f(x)| < \alpha$. By the quantifier swapping trick, we have that for each $\delta > 0$, there is N such that $\Delta_\alpha(f, f_n)$ has measure $\leq \delta$ for all $n \geq N$. Thus, $d_\alpha(f_n, f) \to 0$ as $n \to \infty$. Hence $f_n \to_\mu f$. $\square$

Almost uniform convergence.

Let X be a set and $f_n, f : X \to \mathbb{R}$. Recall that we say that (f_n) converges uniformly to f, denoted $f_n \to_u f$, if $\|f_n - f\|_u \to 0$ as $n \to \infty$, where

$$\|g\|_u := \sup_{x \in X} |g(x)|.$$

Egorov's theorem. Let (X,μ) be a finite measure space and $f_n, f : X \to \mathbb{R}$ measurable. If $f_n \to f$ a.e. then for each $\delta > 0$, $f_n|_{X'} \to_u f|_{X'}$ for a measurable set $X' \subseteq X$ with $\mu(X \setminus X') \leq \delta$.

Proof. We have that $\forall \varepsilon > 0 \quad \forall x \in X \; \exists N$ s.t. $\forall n \geq N \; |f_n - f| < \varepsilon$.

Let $\varepsilon_k \searrow 0$. Then swapping the quantifiers, for each $k \in \mathbb{N}$, we get N_k such that for all $x \in X \setminus X_k$ we have $\forall n \geq N_k \; |f_n - f| < \varepsilon_k$, where X_k is a set of measure $\leq \delta \cdot 2^{-(k+1)}$. Let $X' := \bigcup_{k \in \mathbb{N}} X_k$, so $\mu(X') \leq \sum_{k \in \mathbb{N}} \mu(X_k) \leq \delta$.

Then $\forall k \quad \exists N_k$ such that for all $x \in X \setminus X'$, we have $\forall n \geq N, \; |f_n(x) - f(x)| < \varepsilon_k$.

Thus, $\forall \varepsilon_k \ \exists N_k \ \|f_n|_{X'} - f|_{X'}\| \leq \varepsilon_k$, so $f_n|_{X'} \to_u f|_{X'}$. $\square$

Product measures

Let $(X, \mathcal{I})$ and $(Y, \mathcal{J})$ be measurable spaces. Recall that $\mathcal{I} \otimes \mathcal{J}$ denotes the σ-algebra generated by the "rectangles", i.e. sets of the form $U \times V$, where $U \in \mathcal{I}$ and $V \in \mathcal{J}$.

Theorem. For any measure spaces $(X, \mathcal{I}, \mu)$ and $(Y, \mathcal{J}, \nu)$, there is a measure ρ on $(X \times Y, \mathcal{I} \otimes \mathcal{J})$ such that $\rho(U \times V) = \mu(U) \cdot \nu(V)$ for each rectangle $U \times V$, i.e. $U \in \mathcal{I}$ and $V \in \mathcal{J}$. If μ and ν are σ-finite, this measure is unique and we denote it by $\mu \times \nu$.

Proof. Let $\mathcal{A}$ be the algebra generated by the rectangles and note that each set in $\mathcal{A}$ is a finite disjoint union of rectangles because $(U \times V)^c = U \times V^c \sqcup U^c \times V \sqcup U^c \times V^c$. To prove the theorem it's enough to show that the formula $\rho(U \times V) := \mu(U) \cdot \nu(V)$ defines a premeasure on $\mathcal{A}$ and apply Carathéodory's theorem. As usual, we need to show that ρ on $\mathcal{A}$ is well-defined, i.e. $\bigsqcup_{i \leq n} U_i \times V_i = \bigsqcup_{j \leq k} \tilde{U}_j \times \tilde{V}_j \Rightarrow \rho\left(\bigsqcup_{i \leq n} U_i \times V_i\right) = \rho\left(\bigsqcup_{j \leq k} \tilde{U}_j \times \tilde{V}_j\right)$. We also need to show that ρ is ctbly additive. Both of this would follow if we prove that $U \times V = \bigsqcup_{n \in \mathbb{N}} U_n \times V_n \Rightarrow \mu(U) \cdot \nu(V) = \sum_{n \in \mathbb{N}} \mu(U_n) \cdot \nu(V_n)$. We prove this using MCT twice. Note that $\mathbb{1}_{U \times V} = \sum_{n \in \mathbb{N}} \mathbb{1}_{U_n \times V_n}$ and

$\mathbb{1}_{U\times V}(x,y) = \mathbb{1}_U(x)\cdot\mathbb{1}_V(y)$ so integrating over Y we get: for each $x\in X$:

$$\int_Y \mathbb{1}_U(x)\cdot\mathbb{1}_V(y)\,d\nu(y) = \int_Y \mathbb{1}_{U\times V}(x,y)\,d\nu(y) = \int_Y \sum_{n\in\mathbb{N}} \mathbb{1}_{U_n\times V_n}(x,y)\,d\nu(y) \overset{MCT}{=} \sum_{n\in\mathbb{N}} \int_Y \mathbb{1}_{U_n\times V_n}(x,y)\,d\nu(y)$$

$$\mathbb{1}_U(x)\cdot\nu(V)$$

$$= \sum_{n\in\mathbb{N}} \int_Y \mathbb{1}_{U_n}(x)\cdot\mathbb{1}_{V_n}(y)\,d\nu(y) = \sum_{n\in\mathbb{N}} \mathbb{1}_{U_n}(x)\,\nu(V_n).$$

Now we integrate over X: $\mu(U)\cdot\nu(V) = \nu(V)\cdot\int_X \mathbb{1}_U\,d\mu = \int_X \mathbb{1}_U\cdot\nu(V)\,d\mu =$

$$= \int \sum_{n\in\mathbb{N}} \mathbb{1}_{U_n}\cdot\nu(V_n)\,d\mu \overset{MCT}{=} \sum_{n\in\mathbb{N}} \nu(V_n)\cdot\int_X \mathbb{1}_{U_n}\,d\mu = \sum_{n\in\mathbb{N}} \mu(U_n)\cdot\nu(V_n).$$

$\square$